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On Certain Extremal Problems Concerning Polynomials

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Certain extremal problems concerning polynomials that have restricted ranges with a node are investigated.

I

Let $\{u_i\}_0^n$ be a Chebyshev system on $[a, b]$ and let $U = \text{span}\{u_i\}_0^n$. Let $f, g \in C[a, b]$ satisfy $f > g$. Given $t^* \in [a, b]$ and $c \in (g(t^*), f(t^*))$, denote $K = \{u \in U: g \leqslant u \leqslant f \text{ and } u(t^*) = c\}$. In this paper we discuss certain extremal problems in K (Section II) and their applications to polynomials (Section III for $t^* = a$ and Section IV for $t^* = 0$).

II

In order to describe our basic results we need

DEFINITION. If there exist $u \in K$ and m points

$$a \leqslant t_1 < \cdots < t_m \leqslant b$$

satisfying either

(i) $t^* = t_j$ for some j and

$$u(t_{m-i}) = \begin{cases} f(t_{m-i}), & i = 2k+1 \\ g(t_{m-i}), & i = 2k \end{cases} \qquad (i \neq m-j) \tag{1}$$

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or

(ii) $t^* = t_j$ for some j and

$$u(t_{m-i}) = \begin{cases} f(t_{m-i}), & i = 2k \\ g(t_{m-i}), & i = 2k+1 \end{cases} \qquad (i \neq m-j) \tag{2}$$

then u is said to alternate m times with respect to (g, f) having the node t^*, denoted by $A_1(u) = m$ or $A_2(u) = m$, respectively.

For convenience we write the following lemma which is from [3, p. 61].

LEMMA. *Let $\{u_i\}_0^n$ be a Chebyshev system on $[a, b]$ and $u \in U$. If u has $n+1$ weak sign changes on a set $\{t_1, ..., t_{n+2}\}$, $a \leqslant t_1 < \cdots < t_{n+2} \leqslant b$* [4, p. 260], *then $u = 0$.*

Our basic result, which is an extension of [1, p. 72, Theorem II.10.2], is as follows:

THEOREM 1. *Let $\{u_i\}_0^n$ be a Chebyshev system on $[a, b]$ and let $f, g \in C[a, b]$ such that there exists a polynomial $v \in K$ satisfying $f > v > g$. Then there exists a unique polynomial $\bar{u} \in K$ satisfying $A_1(\bar{u}) = n+1$ and there exists a unique polynomial $\underline{u} \in K$ satisfying $A_2(\underline{u}) = n+1$.*

Proof. The proof of uniqueness may proceed as in Theorem [1, p. 66, II.10.1].

Without loss of generality we assume that $t^* < b$; otherwise for $t^* = b$ we may treat the functions $f^*(T) = f(-T)$ and $g^*(T) = g(-T)$ defined on $[-b, -a]$ and the set $\{u^* \in \operatorname{span}\{u_i(-T)\}_0^n : g^* \leqslant u^* \leqslant f^*, u^*(T^*) = c\}$, in which $T^* = -t^* = -b < -a$.

Take

$$g_k^*(t) = \begin{cases} 0, & t \in [a, b-(b-a)/k] \\ \text{a linear function}, & t \in [b-(b-a)/k, b] \\ 2\,\|f-g\|, & t = b, \end{cases}$$

$$g_k(t) = g(t) - g_k^*(t),$$

$$K_k = \{u \in U : g_k \leqslant u \leqslant f, u(t^*) = c\},$$

$$k = 1, 2, \ldots.$$

Let $v_k \in K_k$ be the best approximation to g_k from K_k, $k = 1, 2, \ldots$. Then for each k there exist $n+2$ points [2]

$$a \leqslant t_1^k < t_2^k < \cdots < t_{n+2}^k \leqslant b$$

such that $t^* = t^k_{j_k}$ and one of the following relations occurs:

$$v_k(t^k_i) = \begin{cases} f(t^k_i) \quad \text{or} \quad g_k(t^k_i) + \|v_k - g_k\|, & i = 2k' + 1 \\ g_k(t^k_i), & i = 2k' \end{cases} \qquad (i \neq j_k) \quad (3)$$

or

$$v_k(t^k_i) = \begin{cases} f(t^k_i) \quad \text{or} \quad g_k(t^k_i) + \|v_k - g_k\|, & i = 2k' \\ g_k(t^k_i), & i = 2k' + 1 \end{cases} \qquad (i \neq j_k). \quad (4)$$

We may assume, selecting a subsequence if necessary, that

(a) $v_k \to u$ as $k \to \infty$ for some $u \in K$;

(b) All v_k satisfy the same one of the above two relations, say (3);

(c) $t^k_i \to t_i$ as $k \to \infty$, $i = 1, ..., n+2$, $t^k_{j_k} = t_j = t^*$, which satisfy that $t_1 \leqslant \cdots \leqslant t_{n+2}$.

Assertion 1. If $t_i = t_{i+1}$ for some $i \leqslant n+1$, then $t_i = t_{i+1} = b$, whence $t_i = \cdots = t_{n+2} = b$.

In fact, suppose on the contrary that $t_i = t_{i+1} < b$. For k large enough we have that $g_k(t^k_i) = g(t^k_i)$ and $g_k(t^k_{i+1}) = g(t^k_{i+1})$. Also it follows from $t_i = t_{i+1}$ that

$$\lim_{k \to \infty} (v_k(t^k_i) - v_k(t^k_{i+1})) = 0,$$

whence by (3) either

$$\lim \|v_k - g_k\| = 0, \qquad j \bar{\in} \{i, i+1\}$$

or

$$\lim \|v_k - g_k\| + g(t_i) - c = 0, \qquad j \in \{i, i+1\}.$$

But

$$\lim \|v_k - g_k\| \geqslant \lim(v_k(b) - g_k(b)) = u(b) - g(b) + 2\,\|f - g\| \geqslant 2\,\|f - g\|, \quad (5)$$

a contradiction.

Assertion 2.

$$t_1 < \cdots < t_{n+1}. \quad (6)$$

By Assertion 1 it suffices to show that $t_n < t_{n+1}$. Suppose not and let $t_n = t_{n+1}$. Then by Assertion 1, $t_n = t_{n+1} = t_{n+2} = b > t^*$. Whence $j < n$ and

$$v_k(t_i^k) = g_k(t_i^k) \qquad \text{for some } i, \quad i \geqslant n+1. \tag{7}$$

For such an index i, by (3) we must have that

$$v_k(t_{i-1}^k) = f(t_{i-1}^k) \qquad \text{or} \qquad g_k(t_{i-1}^k) + \|v_k - g_k\|. \tag{8}$$

Since $\lim(v_k(t_i^k) - v_k(t_{i-1}^k)) = 0$, it follows from (5), (7), and (8) that either

$$0 = \lim(g_k(t_i^k) - f(t_{i-1}^k)) \leqslant \lim(g(t_i^k) - f(t_{i-1}^k)) = g(b) - f(b)$$

or

$$\begin{aligned} 0 &= \lim(g_k(t_i^k) - g_k(t_{i-1}^k) - \|v_k - g_k\|) \\ &= \lim(g(t_i^k) - g(t_{i-1}^k)) + \lim(g_k^*(t_{i-1}^k) - g_k^*(t_i^k)) - \lim \|v_k - g_k\| \\ &\leqslant 0 + 0 - 2\,\|f - g\| \\ &= -2\,\|f - g\|. \end{aligned}$$

In any case it will give a contradiction. Thus $t_n \neq t_{n+1}$.

Assertion 3.

$$j \leqslant n+1. \tag{9}$$

If possible, assume that $j = n+2$, i.e., $t^* = t_{n+2}$. Of course by Assertion 1 and the assumption of $t^* < b$ we have that $t_1 < \cdots < t_{n+1} < t_{n+2} = t^*$. For any $w \in K$ we see that $u - w$ has $n+1$ weak sign changes on a set $\{t_1, ..., t_{n+2}\}$ and $w = u$ by Lemma, a contradiction. The assertion is complete.

By Assertion 3 from (3) and (5) we get that $t^* = t_j$ and

$$u(t_i) = \begin{cases} f(t_i), & i = 2k'+1 \\ g(t_i), & i = 2k' \end{cases} \qquad (1 \leqslant i \leqslant n+1,\ i \neq j),$$

i.e., u satisfies $A_1(u) = n+1$ or $A_2(u) = n+1$. Similarly, if we put that

$$K_k^* = \{u \in U\colon g \leqslant u \leqslant f_k,\ u(t^*) = c\},$$

where $f_k = f + g_k^*$, we can get $u^* \in K$ satisfying $A_1(u^*) = n+1$ or $A_2(u^*) = n+1$.

Assertion 4. $u \neq u^*$.

First, we note that there are $w_1, w_2 \in K$ such that $w_1(b) < w_2(b)$, for otherwise we have that $w(b) = v(b) = u(b)$ for any $w \in K$, where $f(b) > v(b) > g(b)$, which implies that $t_{n+1} \neq b$ and $u - w$ has $n+1$ weak sign changes on a set $\{t_1, ..., t_{n+1}, b\}$. Whence we obtain $w = u$ again, a contradiction. From $w_1(b) < w_2(b)$ it follows that $u(b) \leqslant w_1(b) < w_2(b) \leqslant u^*(b)$, i.e., $u \neq u^*$. By Assertion 4 and uniqueness we see that if $\bar{u} = u$ then $\underline{u} = u^*$ or conversely.

COROLLARY 1. *If the function $f(t)$ in Theorem 1 is a polynomial $u(t)$ and $g = 0$, then there exist a unique representation*

$$u(t) = \bar{u}(t) + \underline{u}^*(t)$$

and a unique representation

$$u(t) = \underline{u}(t) + \bar{u}^*(t),$$

where $\bar{u}$ and $\underline{u}$ are defined in Theorem 1 and $A_1(\bar{u}^*) = A_2(\underline{u}^*) = n+1$ with the value $f(t^*) - c$ at t^* instead of c.

Proof. If f is a polynomial then the function $f(t) - \bar{u}(t)$ obviously satisfies that $A_2(f - \bar{u}) = n+1$ with the value $f(t^*) - c$ at t^*. So, by uniqueness, $f(t) - \bar{u}(t) = \underline{u}^*(t)$. Similarly, we have another representation $f(t) - \underline{u}(t) = \bar{u}^*(t)$.

THEOREM 2. *Assume that the assumptions of Theorem 1 are satisfied and further $\{u_i\}_0^n$ is a Chebyshev system on $[a', b'] \supset [a, b]$. Let $\bar{u}$ and $\underline{u}$ be as defined in Theorem 1 and let $u \in U$ satisfy $g(t) \leqslant u(t) \leqslant f(t)$ for $t \in [a, b]$.*

(a) *If $(-1)^{n+1-j} u(t^*) \leqslant (-1)^{n+1-j} c$ and $u \neq \underline{u}$, then*

$u(t) < \underline{u}(t)$,	$t < a$ or $t > b$,	$n = 2m$
$u(t) > \underline{u}(t)$,	$t < a$,	$n = 2m+1$
$u(t) < \underline{u}(t)$,	$t > b$,	$n = 2m+1$;

(b) *If $(-1)^{n+1-j} u(t^*) \geqslant (-1)^{n+1-j} c$ and $u \neq \bar{u}$, then*

$u(t) > \bar{u}(t)$,	$t < a$ or $t > b$,	$n = 2m$
$u(t) < \bar{u}(t)$,	$t < a$,	$n = 2m+1$
$u(t) > \bar{u}(t)$,	$t > b$,	$n = 2m+1$.

Proof. We present only the proof of (a), the proof of (b) being similar.

Since $A_2(\underline{u}) = n+1$ and

$$\underline{u}(t_j) - u(t_j) = c - u(t^*) \begin{cases} \geqslant 0, & n+1-j=2k \\ \leqslant 0, & n+1-j=2k+1, \end{cases}$$

$$\underline{u}(t_{n+1-i}) - u(t_{n+1-i}) \begin{cases} \geqslant 0, & i=2k \\ \leqslant 0, & i=2k+1. \end{cases}$$

Especially

$$\underline{u}(t_1) - u(t_1) \begin{cases} \geqslant 0, & n=2m \\ \leqslant 0, & n=2m+1 \end{cases}$$

and

$$\underline{u}(t_{n+1}) - u(t_{n+1}) \geqslant 0.$$

Thus if for some $t<a$

$$\underline{u}(t) - u(t) \begin{cases} \leqslant 0, & n=2m \\ \geqslant 0, & n=2m+1, \end{cases}$$

then $\underline{u} - u$ has $n+1$ weak sign changes on a set $\{t, t_1, ..., t_{n+1}\}$ and $\underline{u} = u$ by Lemma. Similarly, $\underline{u}(t) - u(t) \leqslant 0$ for some $t > b$ implies $\underline{u} = u$.

COROLLARY 2. *Let the assumptions of Theorem 2 be satisfied and $u \in K$. If $u \neq \bar{u}$ and $u \neq \underline{u}$, then*

$$\begin{array}{llll} \bar{u}(t) < u(t) < \underline{u}(t), & t<a \quad \text{or} \quad t>b, & & n=2m \\ \underline{u}(t) < u(t) < \bar{u}(t), & t<a, & & n=2m+1 \\ \bar{u}(t) < u(t) < \underline{u}(t), & t>b, & & n=2m+1. \end{array}$$

Proof. From Theorem 2 the corollary is immediate.

COROLLARY 3. *Let the assumptions of Theorem 2 be satisfied with $c \geqslant 0$ and $g = -f < f$. Then for any $u \in K$*

$$|u| \leqslant |u^*|, \qquad t<a \qquad \text{or} \qquad t>b,$$

where

$$u^* = \begin{cases} \underline{u}, & n+1-j=2k \\ \bar{u}, & n+1-j=2k+1, \end{cases}$$

in which equality can occur if and only if

$$u = \begin{cases} u^*, & c > 0 \\ \pm u^*, & c = 0. \end{cases}$$

Proof. Let $n+1-j=2k$. Since $-\bar{u}(t^*) = -c \leqslant c$, $(-1)^{n+1-j}(-\bar{u}(t^*)) \leqslant (-1)^{n+1-j}c$. By Part (a) of Theorem 2 we obtain that

$$-\bar{u}(t) \begin{cases} \leqslant \underline{u}(t), & t<a \quad \text{or} \quad t>b, & n=2m \\ \geqslant \underline{u}(t), & t<a, & n=2m+1 \\ \leqslant \underline{u}(t), & t>b, & n=2m+1. \end{cases}$$

Coupled with Corollary 2 we get that

$$\begin{array}{llll} -\underline{u}(t) \leqslant \bar{u}(t) \leqslant u(t) \leqslant \underline{u}(t), & t<a \quad \text{or} \quad t>b, & n=2m \\ \underline{u}(t) \leqslant u(t) \leqslant \bar{u}(t) \leqslant -\underline{u}(t), & t<a, & n=2m+1 \\ -\underline{u}(t) \leqslant \bar{u}(t) \leqslant u(t) \leqslant \underline{u}(t), & t>b, & n=2m+1. \end{array}$$

In any case we have that

$$|u(t)| \leqslant |\underline{u}(t)|, \qquad t<a \quad \text{or} \quad t>b,$$

in which equality can occur if and only if $u=\underline{u}$ or $u=-\underline{u}=\bar{u}$. But $-\underline{u}=\bar{u}$ if and only if $c=0$. Thus the equality in the above inequality can occur if and only if

$$u = \begin{cases} \underline{u}, & c > 0 \\ \pm \underline{u}, & c = 0. \end{cases}$$

Let $n+1-j=2k+1$. Since $-\underline{u}(t^*) = -c \leqslant c$, $(-1)^{n+1-j}(-\underline{u}(t^*)) \geqslant (-1)^{n+1-j}c$. In the remainder of the proof the same analysis as in the case $n+1-j=2k$ is applicable.

III

As usual $T_n(t)$ denotes the Chebyshev polynomial of degree n of first kind, $n=0, 1, \ldots$.

THEOREM 3. *Let P be a polynomial of degree at most $n \geqslant 1$ such that $|P(t)| \leqslant 1$ for $|t| \leqslant 1$. Let s_0 and s_1 be the smallest values of t in $[-1, 1)$ for which $T_n(s_0)=c$ and $T_n(s_1)=-c$, respectively ($|c|<1$).*

(a) *If* $P(-1) \leqslant c$, *then*

$$P(t) \leqslant T_n(\tfrac{1}{2}(1-s_0)\,t+\tfrac{1}{2}(1+s_0)), \qquad |t|>1, \qquad n=2m$$

$$P(t) \leqslant -T_n(\tfrac{1}{2}(1-s_1)\,t+\tfrac{1}{2}(1+s_1)), \qquad t<-1, \qquad n=2m+1$$

$$P(t) \geqslant -T_n(\tfrac{1}{2}(1-s_1)\,t+\tfrac{1}{2}(1+s_1)), \qquad t>1, \qquad n=2m+1.$$

Any of the equalities occurs for some t if and only if it occurs for any t.

(b) *If* $P(-1) \geqslant c$, *then*

$$P(t) \geqslant -T_n(\tfrac{1}{2}(1-s_1)\,t+\tfrac{1}{2}(1+s_1)), \qquad |t|>1, \qquad n=2m$$

$$P(t) \geqslant T_n(\tfrac{1}{2}(1-s_0)\,t+\tfrac{1}{2}(1+s_0)), \qquad t<-1, \qquad n=2m+1$$

$$P(t) \leqslant T_n(\tfrac{1}{2}(1-s_0)\,t+\tfrac{1}{2}(1+s_0)), \qquad t>1, \qquad n=2m+1.$$

Any of the equalities occurs for some t if and only if it occurs for any t.

Proof. We present the proof of (a), the proof of (b) being similar. For simplicity write $S_i(t) \equiv T_n(\frac{1}{2}(1-s_i)t+\frac{1}{2}(1+s_i))$, $i=0, 1$.

As we know

$$T_n(x_i)=(-1)^{n-i}, \qquad x_i=\cos\frac{n-i}{n}\pi, \qquad i=0, 1, ..., n. \tag{10}$$

By the assumptions of the theorem $-1=x_0<s_0$, $s_1<x_1< \cdots <x_n=1$. Put

$$t_1=x_0 \qquad \text{and} \qquad t_{i+1}=(x_i-\tfrac{1}{2}(1+s_0))/(\tfrac{1}{2}(1-s_0)), \qquad i=1, ..., n.$$

Then $t_1<t_2< \cdots <t_{n+1}$ and

$$S_0(t_{n+1-i})=\begin{cases} T_n(s_0)=c, & i=n \\ T_n(x_{n-i})=(-1)^i, & i=0, ..., n-1, \end{cases}$$

which means $A_2(S_0)=n+1$ with $f=1$, $g=-1$ and $t^*=-1$, i.e., $\underline{u}=S_0$ by Theorem 1. Similarly $\bar{u}=-S_1$. Thus, if $P(-1)\leqslant c$, then $(-1)^n P(-1)\leqslant (-1)^n c$ for $n=2m$ and $(-1)^n P(-1)\geqslant(-1)^n c$ for $n=2m+1$. The results to be desired follow from Theorem 2.

From Theorem 3 the following is immediate by Corollary 2.

COROLLARY 4. *Under the assumptions of Theorem* 3 *if* $P(-1)=c$ *but* $P(t) \not\equiv T_n(\frac{1}{2}(1-s_0)t+\frac{1}{2}(1+s_0))$ *and* $P(t) \not\equiv -T_n(\frac{1}{2}(1-s_1)t+\frac{1}{2}(1+s_1))$, *then*

$$\begin{aligned}
&-T_n(\tfrac{1}{2}(1-s_1)t+\tfrac{1}{2}(1+s_1)) \\
&\qquad < P(t) < T_n(\tfrac{1}{2}(1-s_0)t+\tfrac{1}{2}(1+s_0)), \qquad |t|>1, \qquad n=2m, \\
&-T_n(\tfrac{1}{2}(1-s_1)t+\tfrac{1}{2}(1+s_1)) \\
&\qquad < P(t) < T_n(\tfrac{1}{2}(1-s_0)t+\tfrac{1}{2}(1+s_0)), \qquad t>1, \qquad n=2m+1, \\
&T_n(\tfrac{1}{2}(1-s_0)t+\tfrac{1}{2}(1+s_0)) \\
&\qquad < P(t) < -T(\tfrac{1}{2}(1-s_1)t+\tfrac{1}{2}(1+s_1)), \qquad t<-1, \qquad n=2m+1.
\end{aligned}$$

Remark. Theorem 3 is an extension of the theorem by Rivlin and Shapiro [5], because we have

COROLLARY 5. *Under the assumptions of Theorem* 3, *if* $P(-1)=c$ *with* $0 \leqslant c \leqslant 1$ *then*

$$|P(t)| \leqslant |T_n(\tfrac{1}{2}(1-s)t+\tfrac{1}{2}(1+s))|, \qquad |t|>1, \tag{11}$$

where

$$s=\begin{cases} s_0, & n=2m \\ s_1, & n=2m+1. \end{cases}$$

Equality can occur in (11) *if and only if*

$$\begin{cases} P(t)=\begin{cases} T_n(\frac{1}{2}(1-s)t+\frac{1}{2}(1+s)), & n=2m \\ -T_n(\frac{1}{2}(1-s)t+\frac{1}{2}(1+s)), & n=2m+1 \end{cases} & (c>0) \\ P(t)=\pm T_n(\frac{1}{2}(1-s)t+\frac{1}{2}(1+s)) & (c=0). \end{cases} \tag{12}$$

Proof. In the proof of Theorem 3 we see that

$$\underline{u}(t)=T_n(\tfrac{1}{2}(1-s_0)t+\tfrac{1}{2}(1+s_0)) \quad \text{and} \quad \bar{u}(t)=-T_n(\tfrac{1}{2}(1-s_1)t+\tfrac{1}{2}(1+s_1)).$$

For the case $0 \leqslant c < 1$ the corollary follows directly from Corollary 3 because $j=1$.

For the case $c=1$ we have that $s_0=-1$ for $n=2m$ and $s_1=-1$ for $n=2m+1$, which means that $s=-1$ and $T_n(\frac{1}{2}(1-s)t+\frac{1}{2}(1+s)) \equiv T_n(t)$. Thus the corollary is a well known result.

IV

In this section the main result is

THEOREM 4. *Let P be a polynomial of degree at most $n=2m$ $(m\geqslant 1)$ such that $|P(t)|\leqslant 1$ for $|t|\leqslant 1$ and $P(0)=c$. Let s_0 and s_1 be the smallest values of t in $[0,1)$ for which $T_n(s_0)=c$ and $T_n(s_1)=-c$, respectively. Then*

(a) *For $|c|<1$*

$$-T_n(\sqrt{(1-s_1^2)t^2+s_1^2})\leqslant P(t)\leqslant T_n(\sqrt{(1-s_0^2)t^2+s_0^2}),\qquad |t|>1.$$

Any of the equalities can occur for some t if and only if it occurs for any t.

(b) *For $0\leqslant c\leqslant 1$ and*

$$s=\begin{cases} s_0, & m=2k\\ s_1, & m=2k+1,\end{cases}$$

$$|P(t)|\leqslant |T_n(\sqrt{(1-s^2)t^2+s^2})|,\qquad |t|>1.$$

Equality can occur if and only if

$$\begin{cases} P(t)=\begin{cases} T_n(\sqrt{(1-s^2)t^2+s^2}), & m=2k\\ -T_n(\sqrt{(1-s^2)t^2+s^2}), & m=2k+1\end{cases} & (c>0)\\ P(t)=\pm T_n(\sqrt{(1-s^2)t^2+s^2}) & (c=0).\end{cases}$$

Proof. (a) For simplicity write $S_i(t)\equiv T_n(\sqrt{(1-s_i^2)t^2+s_i^2})$, $i=0,1$. Clearly $S_i(t)$ is symmetric with respect to t. Putting

$$t_{m+1}=0,\qquad -t_{m+1-i}=t_{m+1+i}=\sqrt{(x_{m+i}^2-s_0^2)/(1-s_0^2)},\qquad i=1,\ldots,m,$$

where x_{m+i}'s are defined in (10), we have that $t_1<\cdots<t_{n+1}$ and

$$S_0(t_{m+1+i})=\begin{cases} T_n(s_0)=c, & i=0\\ T_n(x_{m+i})=(-1)^{m-i}, & i=1,\ldots,m.\end{cases}$$

This means by Theorem 1 that $\underline{u}=S_0$. Similarly we can show $\bar{u}=-S_1$. By Corollary 2 we obtain (a).

(b) Noting that $j=m+1$, the conclusion follows directly from Corollary 3 for the case $0\leqslant c<1$.

For the case $c=1$ we have that $s=0$ and $T_n(\sqrt{(1-s^2)t^2+s^2})\equiv T_n(t)$. The corollary is a well known result.

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